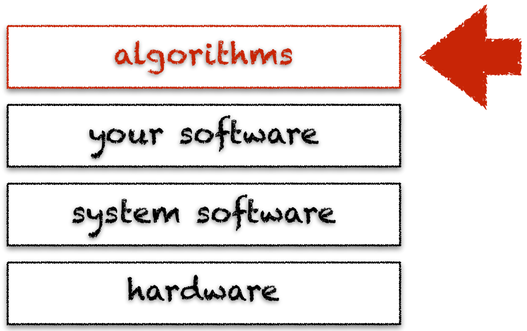

searching
algorithms

# learning objectives




- learn what the searching problem is about
- learn two algorithms for solving this problem
- learn the importance of data structures

# the problem

the searching problems comes in two variants:

- does a collection contain a given element?

$$\exists e \in C \subseteq \mathbb{N} \mid e = 10$$

- what is the value associated with some key in a given associative array ?

$$let\ v \in (k, v) \mid (k, v) \in \mathbb{N} \times \mathbb{R}\ \wedge\ k = 7$$

# sequential search

the simplest searching algorithm based on a brute-force approach

SEQUENTIAL-SEARCH of *key* in $A[1..n]$
**for** $i \leftarrow 1$ **to** $n$
    **if** $A[i] == key$
        **return** *true*
**return** *false*

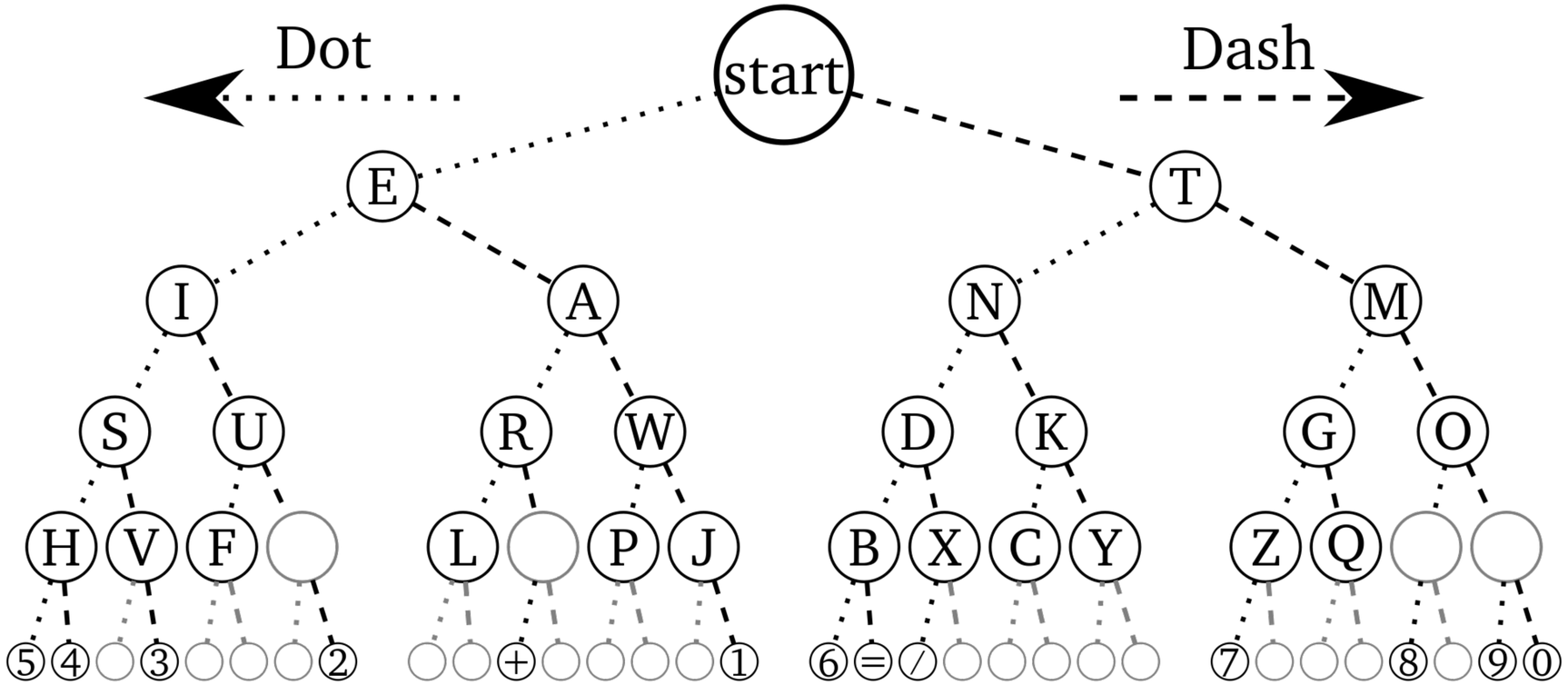
also called linear search

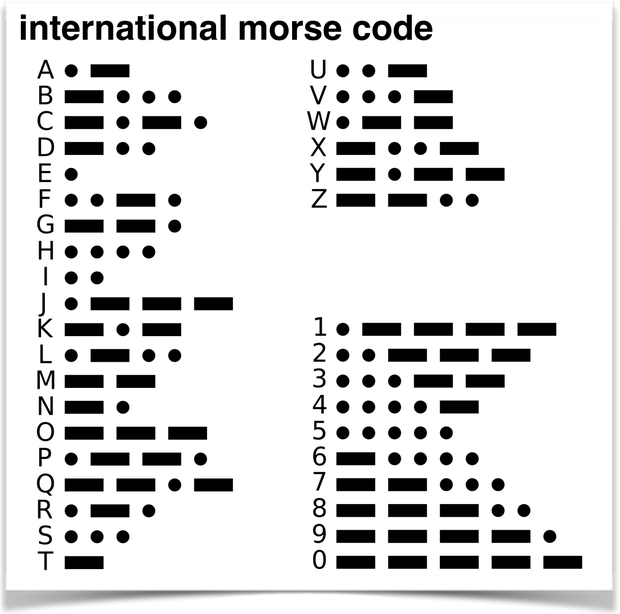
**worst case:** $O(n)$
**best case:** $O(1)$

```
def sequentialSearch(theKey : Int, theArray: Array[Int]) : Boolean = {
  for (key <- theArray)
    if (key == theKey)
      return true
  return false
}
```

# dichotomic search

a dichotomic search consists in selecting between two mutually exclusive alternatives (dichotomies) at each step of the algorithm



# binary search

this algorithm requires a sorted collection

also called half-interval search or logarithmic search

BINARY-SEARCH of $key$ in $A[1..n]$
$low = 1$
$high = n$
**while** $low \leq high$ **do**
    $mid = \left\lfloor \frac{low + high}{2} \right\rfloor$
    **if** $A[mid] > key$ **then**
        $high = mid - 1$
    **else if** $A[mid] < key$ **then**
        $low = mid + 1$
    **else return** *true*
**return** *false*

**worst case:** $O(\log n)$
**best case:** $O(1)$

at each step, it reduces the search space by half by excluding the half that cannot contain the searched key

```
def binarySearch(theKey : Int, theArray: Array[Int]) : Boolean = {
  var low = 0
  var high = theArray.size - 1

  while (low <= high) {
    val mid = (low + high)/2;
    if (theKey < theArray(mid)) {
      high = mid - 1
    } else if (theKey > theArray(mid))
      low = mid + 1
    else
      return true
  }
  return false;
}
```

# binary search

| [0] | [1] | [2] | [3] | [4] | [5] | [6] | [7] | [8] | [9] | [10] | [11] | [12] | [13] | [14] | [15] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 8 | 13 | 17 | 25 | 34 | 41 | 46 | 55 | 63 | **74** | 78 | 81 | 82 | 93 | 99 |

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*mid*

74 > 46

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| | | | | | | | | | | | | | | | |
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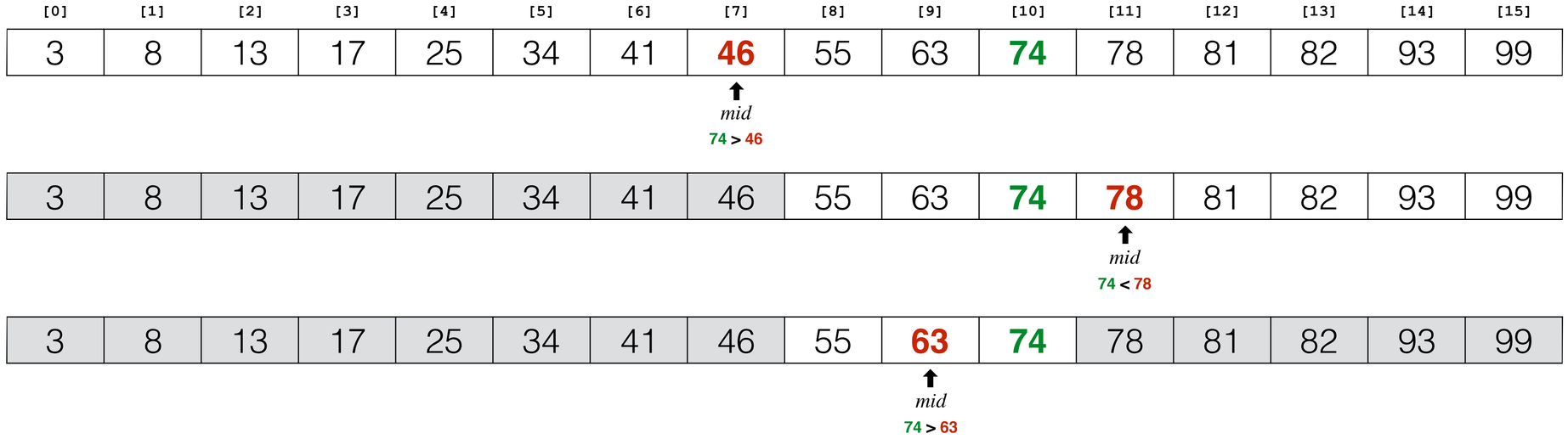
# binary search



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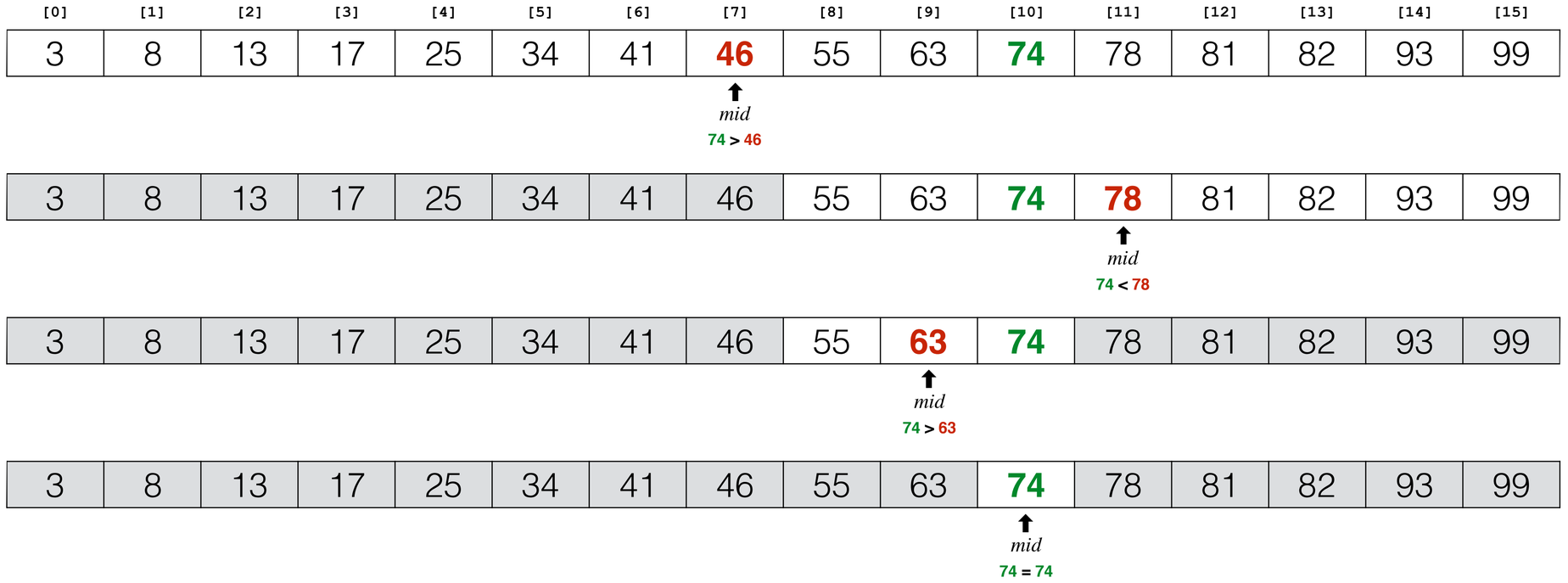
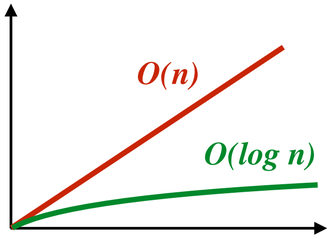
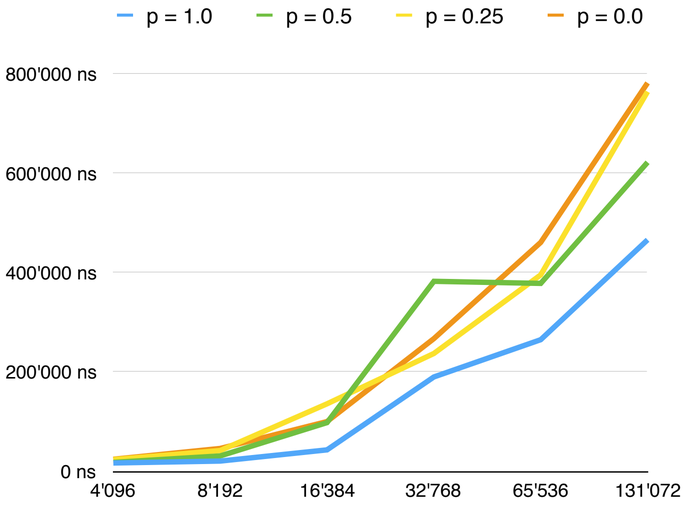


# search performance




## sequential search

| n | p = 1.0 | p = 0.5 | p = 0.25 | p = 0.0 |
|---|---|---|---|---|
| **4'096** | 17'130 ns | 18'306 ns | 23'912 ns | 24'645 ns |
| **8'192** | 20'962 ns | 31'311 ns | 42'125 ns | 46'303 ns |
| **16'384** | 43'458 ns | 98'216 ns | 135'982 ns | 100'311 ns |
| **32'768** | 189'951 ns | 382'093 ns | 237'106 ns | 266'952 ns |
| **65'536** | 264'791 ns | 377'919 ns | 395'050 ns | 460'229 ns |
| **131'072** | 465'458 ns | 621'136 ns | 763'112 ns | 780'680 ns |

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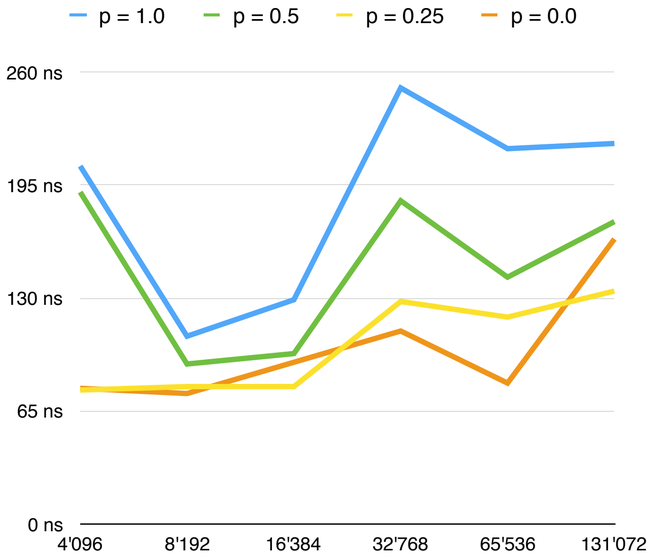

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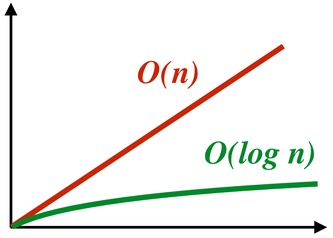


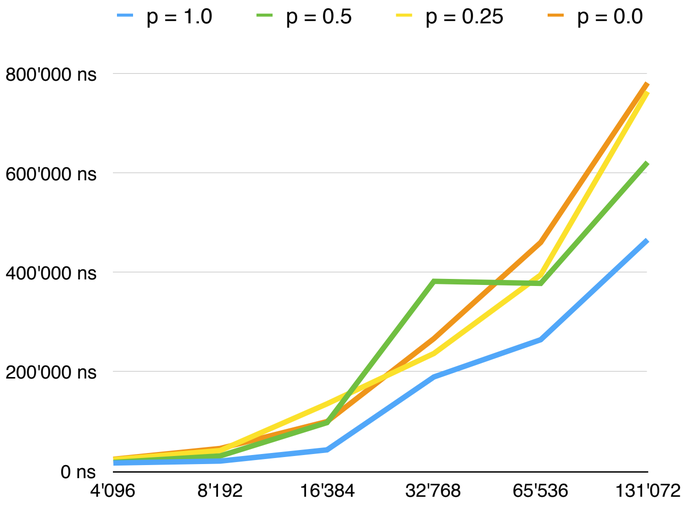


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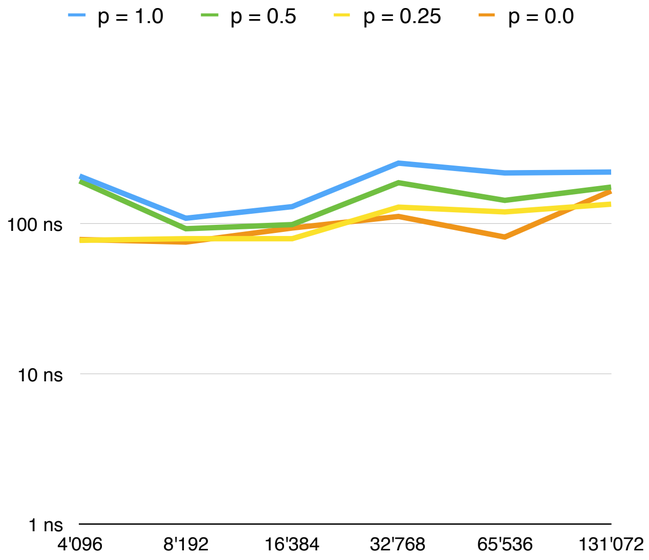

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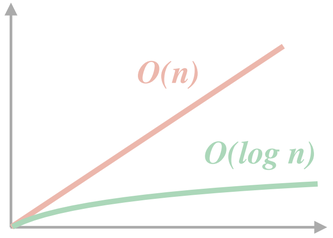



sequential search

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| 32'768 | [illegible] ns | [illegible]'093 ns | [illegible]6 | 266'952 ns |
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where does the "magic" come from?

binary search

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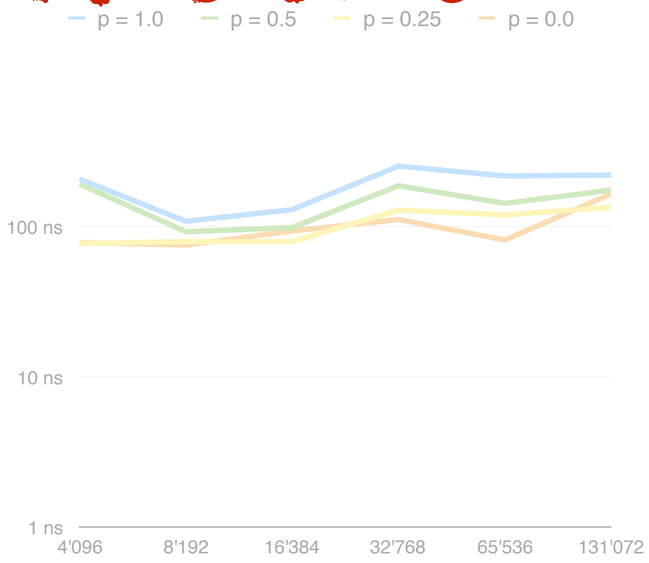

# data structures

the performance of an algorithm often also depends on the data structure

the binary search requires a sorted collection, so part of the cost goes into sorting the collection

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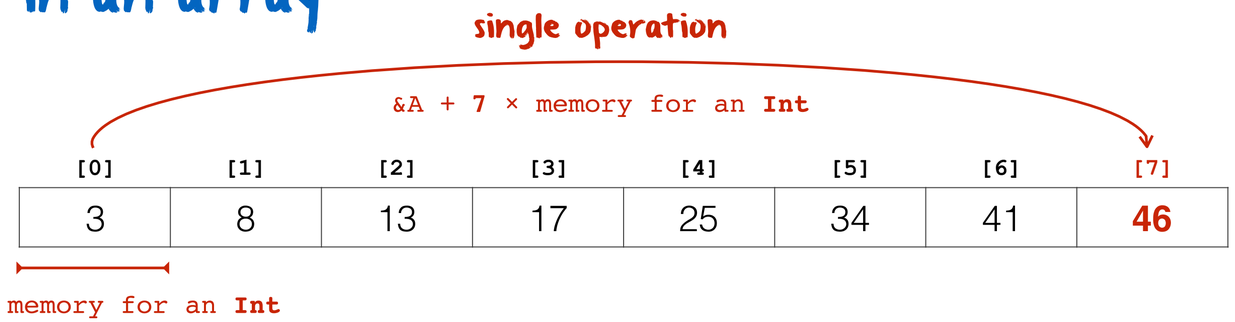
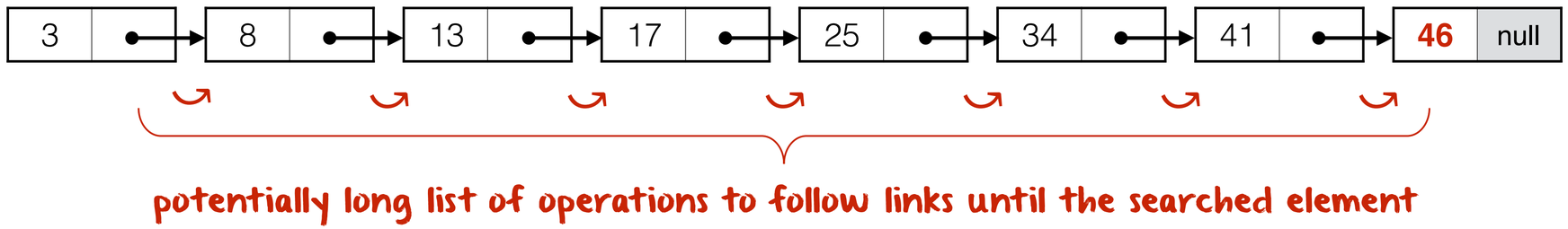
## in an array




accessing a particular element in a collection, say `A[7]`

## in a linked list



# data structures

## sorted array

### remove element

| [0] | [1] | [2] | [3] | [4] | [5] | [6] | [7] |
|---|---|---|---|---|---|---|---|
| 3 | 8 | 13 | 17 | ~~25~~ | 34 | 41 | 46 |
| 3 | 8 | 13 | 17 | 34 | | 41 | 46 |
| 3 | 8 | 13 | 17 | 34 | 41 | | 46 |
| 3 | 8 | 13 | 17 | 34 | 41 | 46 | |

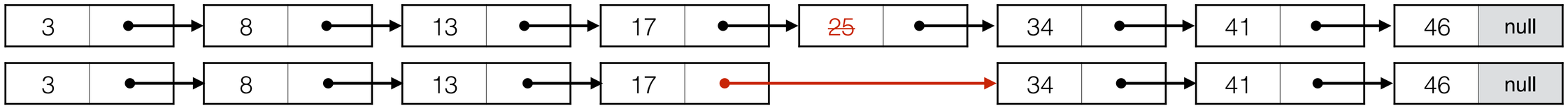
### add element

38

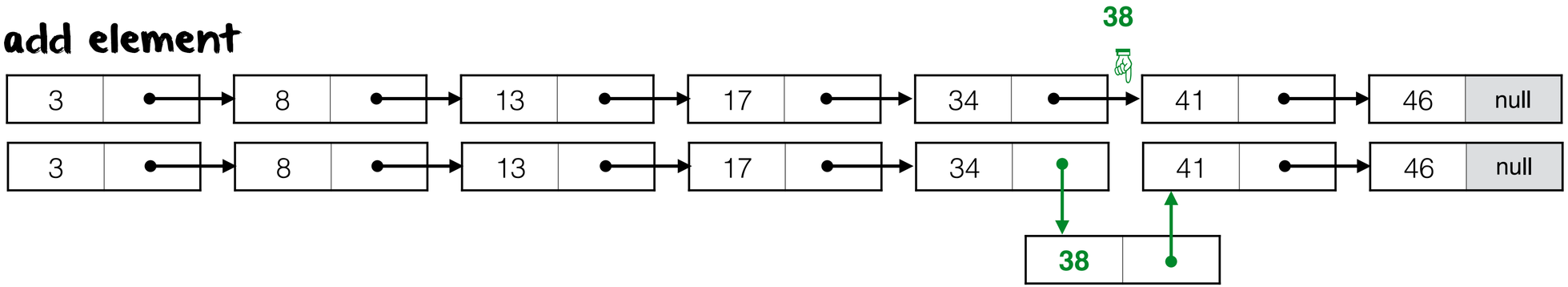
| [0] | [1] | [2] | [3] | [4] | [5] | [6] | [7] |
|---|---|---|---|---|---|---|---|
| 3 | 8 | 13 | 17 | 34 | 41 | 46 | |
| 3 | 8 | 13 | 17 | 34 | 41 | 46 | |
| 3 | 8 | 13 | 17 | 34 | 41 | | 46 |
| 3 | 8 | 13 | 17 | 34 | **38** | 41 | 46 |

## sorted linked list

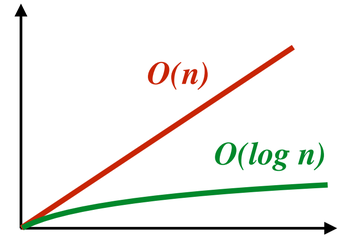
### remove element

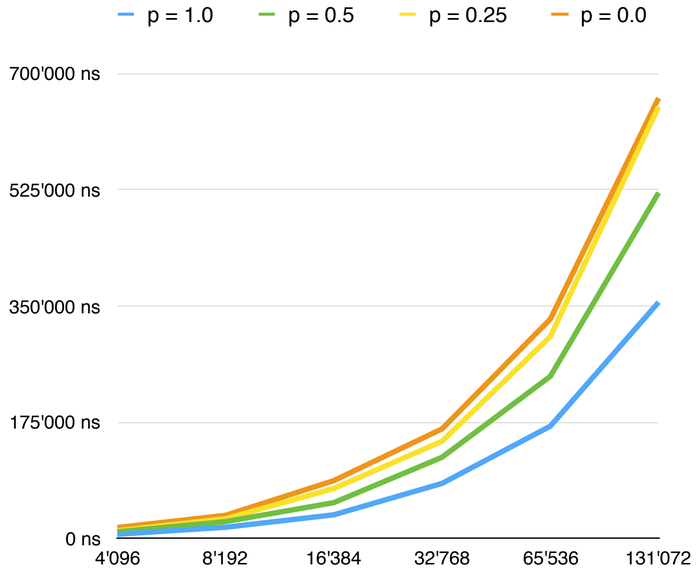



### add element



# search performance



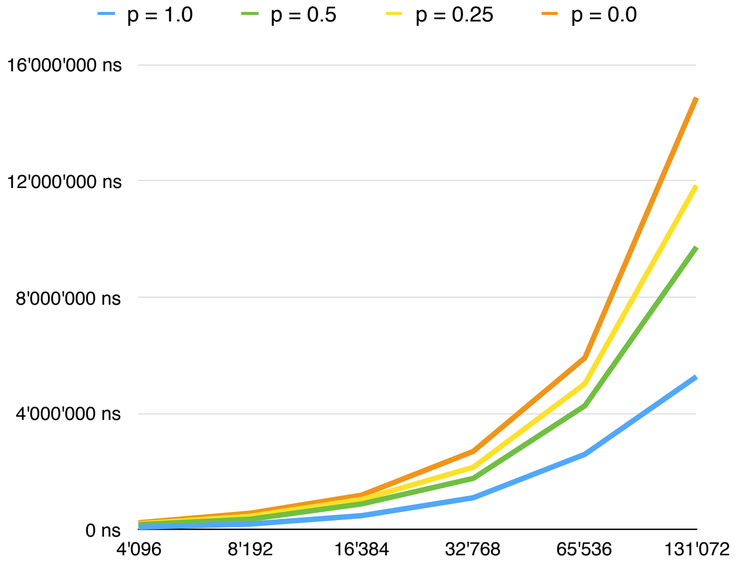




**sequential search in a linked list**

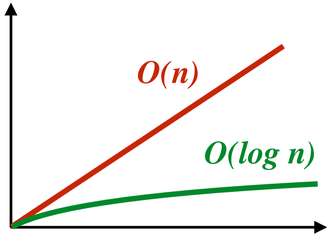
| n | p = 1.0 | p = 0.5 | p = 0.25 | p = 0.0 |
|---|---|---|---|---|
| **4'096** | 6'376 ns | 9'951 ns | 11'516 ns | 16'447 ns |
| **8'192** | 16'865 ns | 25'425 ns | 29'715 ns | 34'780 ns |
| **16'384** | 35'585 ns | 53'881 ns | 75'148 ns | 87'058 ns |
| **32'768** | 82'872 ns | 122'246 ns | 146'411 ns | 164'986 ns |
| **65'536** | 169'044 ns | 244'068 ns | 303'836 ns | 330'198 ns |
| **131'072** | 355'536 ns | 520'393 ns | 649'836 ns | 662'913 ns |

**binary search in a linked list**

| n | p = 1.0 | p = 0.5 | p = 0.25 | p = 0.0 |
|---|---|---|---|---|
| **4'096** | 89'863 ns | 174'655 ns | 200'748 ns | 238'440 ns |
| **8'192** | 193'307 ns | 360'377 ns | 457'862 ns | 565'766 ns |
| **16'384** | 482'478 ns | 886'582 ns | 1'038'265 ns | 1'187'520 ns |
| **32'768** | 1'098'888 ns | 1'765'568 ns | 2'146'730 ns | 2'693'595 ns |
| **65'536** | 2'593'438 ns | 4'261'790 ns | 5'019'401 ns | 5'911'436 ns |
| **131'072** | 5'265'582 ns | 9'724'940 ns | 11'840'437 ns | 14'867'349 ns |



# search performance




**binary search in an array**

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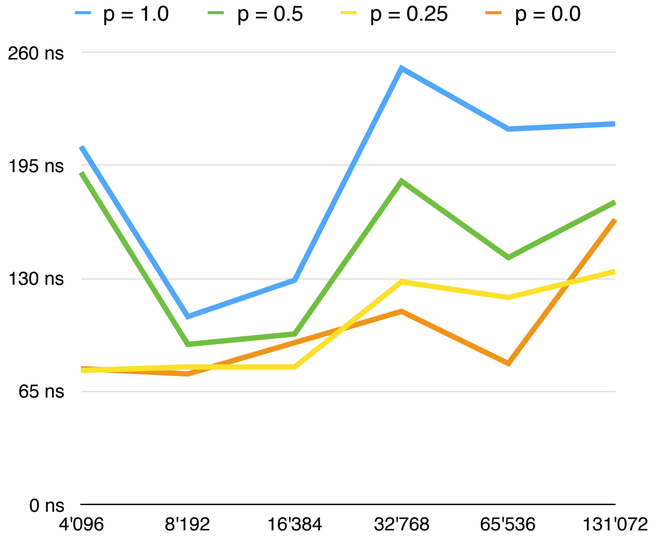


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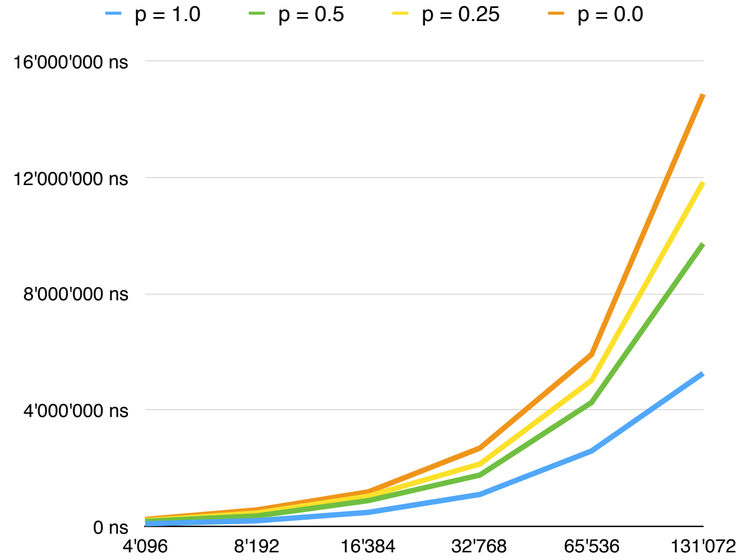
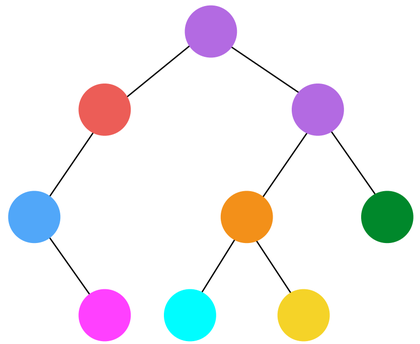


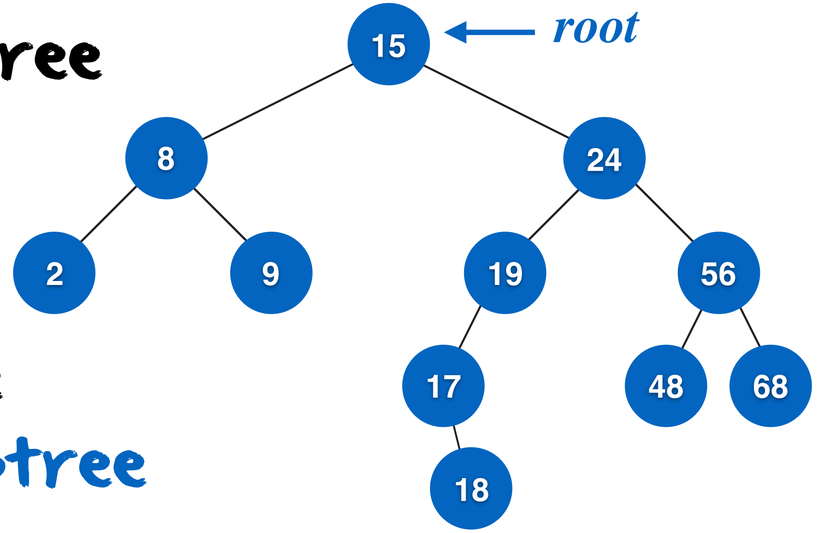
# binary search trees

a binary tree is a tree data structure where each node has at most two children links, which are referred to as the left child and the right child



a binary search tree is a rooted binary tree with the following properties:

- each node has a comparable key
- the key of any node is larger than the keys of all nodes in that node's left subtree
- the key of any node is smaller than the keys of all nodes in that node's right subtree




a subtree is simply the tree that is a child of a node

# binary search trees

in addition, each node might also contain:

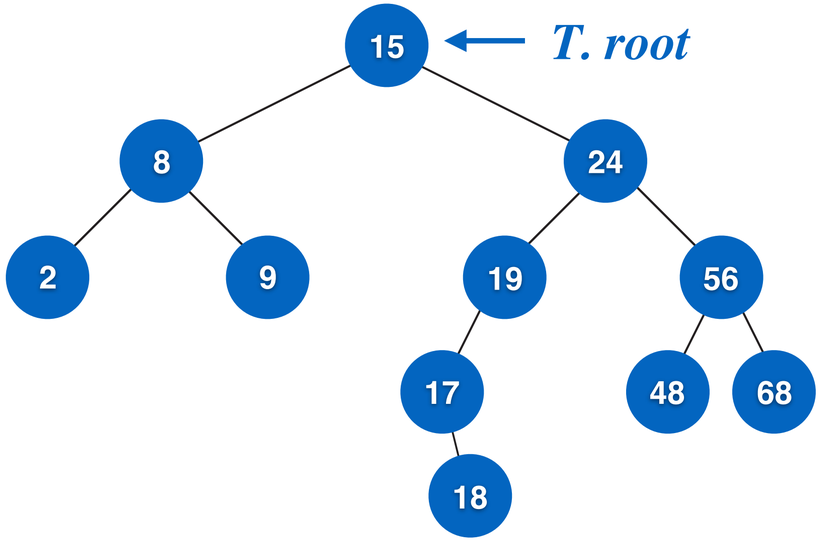
- a value (in the case of associative arrays)
- a link to its parent in the tree, often noted $p$

in full generality, a node of the binary search tree is thus a tuple of the form $( key, value, left, right, p )$

these tuple elements are usually designated as

$x.key$ $x.value$ $x.left$ $x.right$ $x.p$

the tree itself is usually noted $T$ and has a root attribute, noted $T.root$ pointing to the first node of $T$



# binary search trees

## some algorithms

```
INORDER-TREE-WALK(x)
1  if x ≠ NIL
2      INORDER-TREE-WALK(x.left)
3      print x.key
4      INORDER-TREE-WALK(x.right)
```

```
TREE-SEARCH(x, k)
1  if x == NIL or k == x.key
2      return x
3  if k < x.key
4      return TREE-SEARCH(x.left, k)
5  else return TREE-SEARCH(x.right, k)
```

```
ITERATIVE-TREE-SEARCH(x, k)
1  while x ≠ NIL and k ≠ x.key
2      if k < x.key
3          x = x.left
4      else x = x.right
5  return x
```

```
TREE-MINIMUM(x)
1  while x.left ≠ NIL
2      x = x.left
3  return x
```

no need to compare keys!

```
TREE-MAXIMUM(x)
1  while x.right ≠ NIL
2      x = x.right
3  return x
```

```
TREE-SUCCESSOR(x)
1  if x.right ≠ NIL
2      return TREE-MINIMUM(x.right)
3  y = x.p
4  while y ≠ NIL and x == y.right
5      x = y
6      y = y.p
7  return y
```

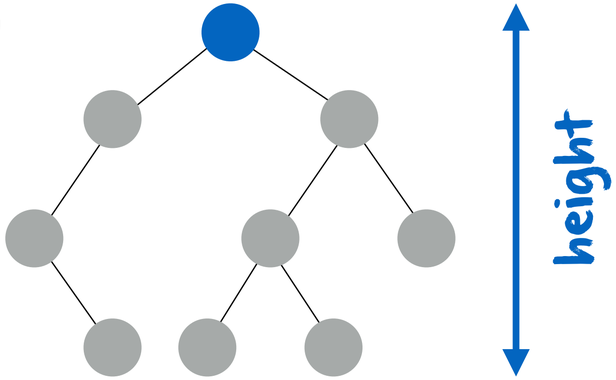
```
TREE-INSERT(T, z)
1   y = NIL
2   x = T.root
3   while x ≠ NIL
4       y = x
5       if z.key < x.key
6           x = x.left
7       else x = x.right
8   z.p = y
9   if y == NIL
10      T.root = z
11  elseif z.key < y.key
12      y.left = z
13  else y.right = z
```

tree was empty

from: ***Introduction to Algorithms***
*by T. H. Cormen et al.*
***3rd Edition***
*MIT Press, 2009*

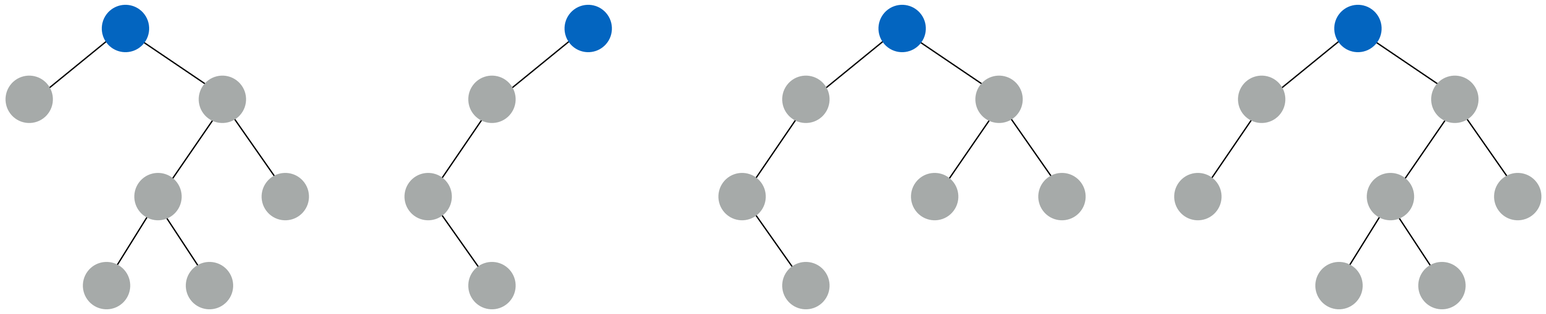
# balanced trees

the height of a tree is the maximum distance of any node from the root in terms of number of edges to traverse




a height-balanced (or simply balanced) tree is a tree whose subtrees have the following properties:

- they differ in height by no more than one
- they are height-balanced as well



why is it interesting to use a balanced binary search tree?